Schrödinger Bridge Problem

Bangyan Liao
liaobangyan@westlake.edu.cn

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WESTLAKE UNIVERSITY

Outline

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Theorem (Girsanov Theorem)

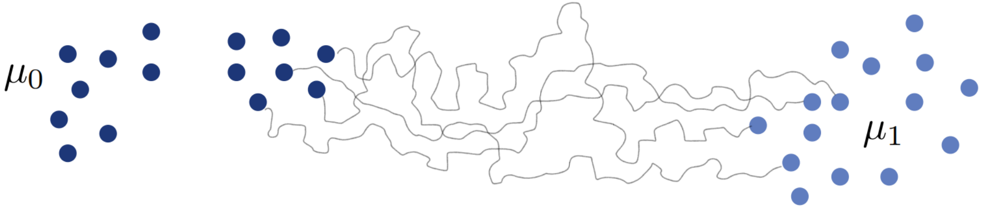
Given two Itô processes with the same constant volatility: $d\mathbf{x}(t) = \mathbf{b}_1(t) + \sigma\, d\beta(t), \quad \mathbf{x} = \mathbf{x}_0$ *and* $d\mathbf{y}(t) = \mathbf{b}_2(t) + \sigma\, d\beta(t), \quad \mathbf{y} = \mathbf{x}_0$*, the RN derivative of their respective path measures* $\mathbb{P}, \mathbb{Q}$ *is given by*

$$\frac{d\mathbb{P}}{d\mathbb{Q}}(\cdot) = \exp\left(-\frac{1}{2\sigma^2}\int_0^t \|\mathbf{b}_1(s) - \mathbf{b}_2(s)\|^2\, ds + \frac{1}{\sigma^2}\int_0^t (\mathbf{b}_1(s) - \mathbf{b}_2(s))^\top\, d\beta(s)\right) \quad (1)$$

where the type signature of this RN derivative is $\frac{d\mathbb{P}}{d\mathbb{Q}} : C(T, \mathbb{R}^d) \to \mathbb{R}$.

Definition (Path Measure)

For an Itô process of the form $dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dW_t$ defined in $[0, T]$, we call $\mathbb{P}$ the path measure of the above process, with outcome space $\Omega = C([0, T], \mathbb{R}^d)$, if the distribution $\mathbb{P}$ describes a weak solution to the above SDE.



Let us define a forward process X_t that solves $dX_t = \mu_+(t, X_t)\,dt + \sigma(t, X_t)\,dW_t$ and a backward process $X_{\tilde{t}}$ that solves $dX_{\tilde{t}} = \mu_-(\tilde{t}, X_{\tilde{t}})\,d\tilde{t} + \sigma(\tilde{t}, X_{\tilde{t}})\,dW_{\tilde{t}}$. We can also define the corresponding probability measure as $p_t(x)$ and $p_{\tilde{t}}(x)$ respectively. Then, if $p_{T-t}(x) = p_{\tilde{t}}(x)$. The Nelson's Duality tells us that

$$\mu_+(t, x) - \mu_-(\tilde{t}, x) = \sigma^2 \nabla_x \log p_t(x)$$
$$\mu_-(\tilde{t}, x) - \mu_+(t, x) = \sigma^2 \nabla_x \log p_{\tilde{t}}(x)$$

(2)

Theorem (Brenier-Benamou Formulation (Eularian Formalism))

$$
\begin{aligned}
\inf_{(\mu,\nu)} \quad & \int_0^1 \int_{\mathbb{R}^n} \frac{1}{2}\|\nu(t,x)\|^2 \,\mathrm{d}\mu_t(x)\,\mathrm{d}t \\
s.t. \quad & \frac{\partial \mu_t}{\partial t} + \nabla \cdot (\nu \mu_t) = 0, \\
& \mu_{t=0} = \mu_0, \quad \mu_{t=1} = \mu_1,
\end{aligned} \tag{3}
$$

Theorem (SOC Formulation (Lagrangian Formalism))

$$
\begin{aligned}
\inf_{\nu} \quad & \mathbb{E}\left\{ \int_0^1 \frac{1}{2}\|\nu(t,X_t)\|^2 \,\mathrm{d}t \right\} \\
s.t. \quad & \mathrm{d}X_t = \nu(t,X_t)\,\mathrm{d}t, \\
& X_0 \sim \mu_0, \quad X_1 \sim \mu_1,
\end{aligned} \tag{4}
$$

Proof. (From OT to Brenier-Benamou Formulation).

Please refer the Sec. 3.3 in Stochastic control liaisons: Richard sinkhorn meets gaspard monge on a schrödinger bridge. □

Proof. (From Brenier-Benamou to SOC Formulation).

Notice that

$$\mathbb{E}_x \left\{ \int_0^1 \frac{1}{2} \|\nu(t, X_t(x))\|^2 \, \mathrm{d}t \right\} = \int_0^1 \int_{\mathbb{R}^n} \frac{1}{2} \|\nu(t, X_t(x))\|^2 \, \mathrm{d}\mu_0(x) \, \mathrm{d}t \tag{5}$$

Then, by applying the definition of push-forward operator $X_{t\#}$

$$\int f(X_t(x)) d\mu_0(x) = \int f(x) d\mu_t(x) \tag{6}$$

we can get the equivalent transformation. □

Optimality Condition for SOC-OT

SOC Perspective of OT

Theorem (Optimality Condition for SOC-OT)

Let $\mu_t^*(x)$ *with* $t \in [0,1]$ *and* $x \in \mathbb{R}^n$, *satisfy*

$$\frac{\partial \mu_t^*}{\partial t} + \nabla \cdot (\mu_t^* \nabla \lambda) = 0, \quad \mu_{t=0}^* = \mu_0, \tag{7}$$

where λ *is a solution of the Hamilton-Jacobi equation*

$$\frac{\partial \lambda}{\partial t} + \frac{1}{2}\|\nabla \lambda\|^2 = 0 \tag{8}$$

for some boundary condition $\lambda(1,x) = \lambda_1(x)$. *If* $\mu_{t=1}^* = \mu_1$, *then the pair* (μ^*, ν^*) *with* $\nu^*(t,x) = \nabla \lambda(t,x)$ *is the solution.*

Proof. (Optimality Condition for SOC-OT).

Consider the unconstrained minimization of the Lagrangian

$$\mathcal{L}(\mu,\nu) = \int_0^1 \int_{\mathbb{R}^n} \left[\frac{1}{2} \|\nu(t,x)\|^2 \mu_t(x) + \lambda(t,x) \left(\frac{\partial \mu_t}{\partial t} + \nabla \cdot (\nu \mu_t) \right) \right] \mathrm{d}x \, \mathrm{d}t \quad (9)$$

where μ_t satisfies the boundary condition. Then, integrating by parts, assuming that limits for $\|x\| \to \infty$ are zero, we get

$$\begin{aligned} &\int_0^1 \int_{\mathbb{R}^n} \left[\frac{1}{2} \|\nu(t,x)\|^2 + \left(-\frac{\partial \lambda}{\partial t} - \nabla \lambda \cdot \nu \right) \right] \mu_t(x) \mathrm{d}x \, \mathrm{d}t \\ &\quad + \int_{\mathbb{R}^n} \int_0^1 \frac{\partial \lambda(t,x) \mu_t(x)}{\partial t} \mathrm{d}t \, \mathrm{d}x + \int_0^1 \int_{\mathbb{R}^n} \frac{\partial \lambda(t,x) \nu(t,x) \mu_t(x)}{\partial x} \mathrm{d}x \, \mathrm{d}t \end{aligned} \quad (10)$$

Proof. (Optimality Condition for SOC-OT).

The last two integrals are constant for a fixed λ and can therefore be discarded. Then, we consider doing this in two stages, starting from minimization with respect to ν for a fixed flow of probability densities μ_t. Pointwise minimization of the integral at each time gives that

$$\nu^*_{\mu_t}(t, x) = \nabla \lambda(t, x) \tag{9}$$

Then, substituting this expression for the optimal control, we obtain

$$J(\mu) = -\int_{\mathbb{R}^n} \int_0^1 \left[\frac{\partial \lambda}{\partial t} + \frac{1}{2} \|\nabla \lambda\|^2 \right] \mu_t(x) \mathrm{d}t \, \mathrm{d}x \tag{10}$$

In view of this, if λ satisfies the Hamilton-Jacobi equation $\frac{\partial \lambda}{\partial t} + \frac{1}{2} \|\nabla \lambda\|^2 = 0$, then $J(\mu)$ is identically zero. □

The generalization to non-trivial underlying dynamics of the form $\dot{x} = f(t,x) + \nu$ leads in a similar manner to

Theorem (SOC with Prior Drift (Eularian Formalism))

$$\begin{aligned} \inf_{(\mu,\nu)} \quad & \int_0^1 \int_{\mathbb{R}^n} \frac{1}{2} \|\nu(t,x) - f(t,x)\|^2 \, \mathrm{d}\mu_t(x) \, \mathrm{d}t \\ s.t. \quad & \frac{\partial \mu_t}{\partial t} + \nabla \cdot (\nu \mu_t) = 0, \\ & \mu_{t=0} = \mu_0, \quad \mu_{t=1} = \mu_1, \end{aligned} \tag{11}$$

Theorem (SOC with Prior Drift (Lagrangian Formalism))

$$\begin{aligned} \inf_{\nu} \quad & \mathbb{E} \left\{ \int_0^1 \frac{1}{2} \|\nu(t, X_t)\|^2 \, \mathrm{d}t \right\} \\ s.t. \quad & \mathrm{d}X_t = (f(t, X_t) + \nu(t, X_t)) \, \mathrm{d}t, \qquad X_0 \sim \mu_0, \quad X_1 \sim \mu_1, \end{aligned} \tag{12}$$

The generalization to non-trivial underlying dynamics of the form $\dot{x} = f(t, x) + \nu$ leads in a similar manner to

Theorem (Optimality Condition for SOC-OT with prior drift)

If λ satisfies the Hamilton-Jacobi equation

$$\frac{\partial \lambda}{\partial t} + f \cdot \nabla \lambda + \frac{1}{2} \|\nabla \lambda\|^2 \tag{13}$$

and is such that the solution μ^* to

$$\frac{\partial \mu_t^*}{\partial t} + \nabla \cdot [(f + \nabla \lambda) \mu_t^*] = 0, \quad \mu_{t=0}^* = \mu_0, \tag{14}$$

satisfies the end-point condition $\mu_{t=1}^* = \mu_1$ as well, then the pair $(\mu_t^*, \nu_t^* = f_t + \nabla \lambda)$ is the solution, provided $\lambda \mu_t^*$ vanishes as $\|x\| \to \infty$ for each fixed t.

Outline

Definition (Dynamic Schrödinger Bridge Problem)

$$P_{\text{SBP}} := arg \min_{P \in \mathcal{D}(\rho_0, \rho_1)} \mathbb{D}(P||W^{\varepsilon}) \tag{15}$$

where W^{ε} represents the prior path measure induced by the Wiener process $\mathrm{d}X = \sqrt{\varepsilon}\mathrm{d}W$ and

$$\mathbb{D}(P||Q) = \mathbb{E}_P \left\{ \log \frac{\mathrm{d}P}{\mathrm{d}Q} \right\}, \quad \text{if} P \ll Q \tag{16}$$

denotes the relative entropy (KL divergence), and

$$\mathcal{D}(\rho_0, \rho_1) = \{P \in \mathcal{C}([0,1], \mathbb{R}^n) | P_{t=0} = \rho_0, P_{t=1} = \rho_1\} \tag{17}$$

denotes a path measure has marginal measure ρ_0 and ρ_1 at time $t = 0$ and $t = 1$, respectively.

Definition (Static Schrödinger Bridge Problem)

$$\{P_{\text{SBP}}\}_{01} := arg \min_{P_{01} \in \Pi(\rho_0, \rho_1)} \mathbb{D}(P_{01} || W^{\varepsilon}_{01}) \tag{18}$$

where W^{ε}_{01} represents the Wiener process induced prior path measure marginalized at time $t = 0$ and $t = 1$. Besides, the set of product measure defines as

$$\Pi(\rho_0, \rho_1) = \left\{ P_{01} : \mathbb{R}^n \times \mathbb{R}^n \to [0, 1] | \int_y \mathrm{d}P_{01}(x, y) = \rho_0(x), \int_x \mathrm{d}P_{01}(x, y) = \rho_1(y) \right\} \tag{19}$$

Proof. (From Dynamic SBP to Static SBP).

By applying the disintegration theorem

$$
\begin{aligned}
\mathbb{D}(P||W^{\varepsilon}) &= \int \log\left(\frac{\mathrm{d}P}{\mathrm{d}W^{\varepsilon}}\right)\mathrm{d}P = \int_{01}\int_{\cdot|01} \log\left(\frac{\mathrm{d}P_{01}}{\mathrm{d}W^{\varepsilon}_{01}}\frac{\mathrm{d}P_{\cdot|01}}{\mathrm{d}W^{\varepsilon}_{\cdot|01}}\right)\mathrm{d}P_{01}\mathrm{d}P_{\cdot|01} \\
&= \int_{01}\int_{\cdot|01} \log\left(\frac{\mathrm{d}P_{01}}{\mathrm{d}W^{\varepsilon}_{01}}\right)\mathrm{d}P_{\cdot|01}\mathrm{d}P_{01} + \int_{01}\int_{\cdot|01} \log\left(\frac{\mathrm{d}P_{\cdot|01}}{\mathrm{d}W^{\varepsilon}_{\cdot|01}}\right)\mathrm{d}P_{\cdot|01}\mathrm{d}P_{01} \\
&= \int_{01} \log\left(\frac{\mathrm{d}P_{01}}{\mathrm{d}W^{\varepsilon}_{01}}\right)\mathrm{d}P_{01} + \int_{01}\int_{\cdot|01} \log\left(\frac{\mathrm{d}P_{\cdot|01}}{\mathrm{d}W^{\varepsilon}_{\cdot|01}}\right)\mathrm{d}P_{\cdot|01}\mathrm{d}P_{01}
\end{aligned}
\tag{20}
$$

Proof. (From Dynamic SBP to Static SBP).

notice that $\mathrm{d}P_{\cdot|01} = \mathrm{d}W^{\varepsilon}_{\cdot|01}$ realizes the so-called Brownian Bridge which defined as

$$\begin{aligned} \mathrm{d}X_t &= \frac{1}{1-t}(x_1 - X_t)\mathrm{d}t + \sqrt{\varepsilon}\mathrm{d}W, \quad X_{t=0} = x_0 \\ P(X_t|X_0) &= N((1-t)x_0 + tx_1, t(1-t)) \end{aligned} \tag{20}$$

After canceling out the last term in the dynamic SBP, we can complete the proof. □

Remarks.

- For simplicity, we can represent the path measure P as a distribution which evolves according to the solution of an SDE of the form

$$\mathrm{d}X_t = v_t \,\mathrm{d}t + \sqrt{\varepsilon}\mathrm{d}W \tag{21}$$

- The disintegration theorem tells us that, if we have the optimal dynamic path measure P^*, then the static path measure P^*_{01} is just the start-end time marginal of the dynamic path measure. If we have the static path measure P^*_{01}, then we can always infer the dynamic path measure by applying the Brownian bridge.

Corollary (EntropicOT-SBP)

The SBP has a close connection in the optimal transport community, where the static SBP is actually equivalent to the entropic optimal transport problem as

$$\min_{\pi \in \Pi(\rho_0, \rho_1)} \int \int \frac{\|x - y\|^2}{2\varepsilon} \mathrm{d}\pi(x, y) + \int \int \log \pi(x, y) \mathrm{d}\pi(x, y) \tag{22}$$

Proof. (EntropicOT-SBP).

Let us define the W_{01}^{ε} as a decomposition $\mathrm{d}W_{01}^{\varepsilon}(x, y) = \mathrm{d}q_0(x)N(y|x, \varepsilon)$. Then,

$$\mathbb{D}(P_{01}||W_{01}^{\varepsilon}) = \int_{01} \log\left(\frac{\mathrm{d}P_{01}}{\mathrm{d}W_{01}^{\varepsilon}}\right)\mathrm{d}P_{01} = \int_{01} (\log \mathrm{d}P_{01})\ \mathrm{d}P_{01} + \int_{01} (\log \mathrm{d}W_{01}^{\varepsilon})\ \mathrm{d}P_{01}$$
$$= \int_{01} (\log \mathrm{d}P_{01})\ \mathrm{d}P_{01} - \int_{01} (\log \mathrm{d}q_0(x))\ \mathrm{d}P_{01} - \int_{01} -\frac{\|x-y\|^2}{2\varepsilon}\mathrm{d}P_{01}$$

$$\min \mathbb{D}(P_{01}||W_{01}^{\varepsilon}) = \min \int\int (\log \mathrm{d}P_{01})\ \mathrm{d}P_{01} + \int\int \frac{\|x-y\|^2}{2\varepsilon}\mathrm{d}P_{01} + \text{const} \tag{23}$$

Let the π represents the P_{01}, which completes the proof. □

Corollary (SOC-SBP)

Besides the entropic OT perspective, we can also view the dynamic SBP from the stochastic optimal control perspective

$$\begin{aligned} &\inf_{v} \mathbb{E}\left\{\int_0^1 \frac{1}{2\varepsilon}\|v(t,X_t)\|^2\,\mathrm{d}t\right\} \\ &\textit{s.t.} \quad \mathrm{d}X_t = v(t,X_t)\,\mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}W_t, \quad X_0 \sim \mu_0, \quad X_1 \sim \mu_1, \end{aligned} \tag{24}$$

Proof. (SOC-SBP).

By applying the Girsanov Theorem,

$$\frac{\mathrm{d}P}{\mathrm{d}W^{\varepsilon}}(\cdot) = \exp\left(\frac{1}{2\varepsilon}\int_0^1 \|v_t(\cdot)\|^2 \,\mathrm{d}t + \frac{1}{\varepsilon}\int_0^1 v_t(\cdot)^\top \,\mathrm{d}W_t\right) \tag{25}$$

we have that

$$\begin{aligned}
\mathbb{D}(P||W^{\varepsilon}) &= \int \log\left(\frac{\mathrm{d}P}{\mathrm{d}W^{\varepsilon}}\right)\mathrm{d}P = \int \log\left(\frac{\mathrm{d}P_0}{\mathrm{d}W_0^{\varepsilon}}\frac{\mathrm{d}P_{\cdot|0}}{\mathrm{d}W_{\cdot|0}^{\varepsilon}}\right)\mathrm{d}P \\
&= \int \log\left(\frac{\mathrm{d}P_0}{\mathrm{d}W_0^{\varepsilon}}\right)\mathrm{d}P_0 + \int \frac{1}{2\varepsilon}\int_0^1 \|v_t(\cdot)\|^2 \,\mathrm{d}t + \frac{1}{\varepsilon}\int_0^1 v_t(\cdot)^\top \,\mathrm{d}W_t\mathrm{d}P \\
&= \mathbb{D}(P_0||W_0^{\varepsilon}) + \int \frac{1}{2\varepsilon}\int_0^1 \|v_t(\cdot)\|^2 \,\mathrm{d}t + \frac{1}{\varepsilon}\int_0^1 v_t(\cdot)^\top \,\mathrm{d}W_t\mathrm{d}P
\end{aligned} \tag{26}$$

Proof. (SOC-SBP).

Since

$$\mathbb{E}\left[\int_0^1 v_t(\cdot)^\top \mathrm{d}W_t\right] = 0, \tag{25}$$

then

$$arg\min \mathbb{D}(P_{01}||W_{01}^{\varepsilon}) = arg\min \mathbb{E}\left[\frac{1}{2\varepsilon}\int_0^1 \|v_t(\cdot)\|^2 \mathrm{d}t\right] \tag{26}$$

which completes the proof. □

Corollary (FD-SBP)

The stochastic optimal control perspective of SBP leads an equivalent fluid dynamic perspective.

$$
\begin{aligned}
&\min_{(\mu_t, v)} \int_0^1 \int_{\mathbb{R}^n} \frac{1}{2\varepsilon} \|v(t,x)\|^2 \, \mathrm{d}\mu_t(x) \, \mathrm{d}t, \\
&s.t. \ \frac{\partial \mu_t}{\partial t} + \nabla \cdot (v \mu_t) - \frac{\varepsilon}{2} \Delta \mu_t = 0, \quad \mu_{t=0} = \mu_0, \ \mu_{t=1} = \mu_1,
\end{aligned} \tag{27}
$$

Corollary (DynamicEOT-SBP)

We can also present a dynamic version for entropic optimal transport SBP as

$$\begin{aligned} &\inf_{(\mu_t, v)} \int_0^1 \int_{\mathbb{R}^n} \left[\frac{1}{2\varepsilon} \|v(t,x)\|^2 + \frac{\varepsilon}{8} \|\nabla log \mu_t\|^2 \right] \mathrm{d}\mu_t(x)\, \mathrm{d}t, \\ &s.t. \frac{\partial \mu_t}{\partial t} + \nabla \cdot (v \mu_t) = 0, \quad \mu_{t=0} = \mu_0,\ \mu_{t=1} = \mu_1, \end{aligned} \tag{28}$$

Outline

Theorem (Optimality Condition of SBP (SOC))

In the following, we give the optimality condition for SBP.

$$\begin{aligned} \frac{\partial V_t}{\partial t} - \frac{\varepsilon}{2}\|\nabla V_t\|^2 + \frac{\varepsilon}{2}\Delta V_t = 0 \\ \frac{\partial \mu_t}{\partial t} + \nabla \cdot (v_t \mu_t) - \frac{\varepsilon}{2}\Delta \mu_t = 0 \end{aligned} \tag{29}$$

where the optimal policy (control)

$$v_t^* = -\sqrt{\varepsilon}\,\nabla V_t(X_t) \tag{30}$$

Proof.

Recall the Itô Lemma for SDE $\mathrm{d}X_t = u(X_t, t)\,\mathrm{d}t + \sqrt{\varepsilon}\mathrm{d}W_t$:

$$\begin{aligned} \mathrm{d}V(X_t, t) &= \frac{\partial V(X_t, t)}{\partial t}\mathrm{d}t + \frac{\partial V(X_t, t)}{\partial x}\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2 V(X_t, t)}{\partial x^2}(\mathrm{d}X_t)^2 \\ &= \frac{\partial V_t}{\partial t}\mathrm{d}t + \mathcal{L}V(x, t)\mathrm{d}t + \nabla V_t(x) \cdot \sqrt{\varepsilon}\mathrm{d}W_t \end{aligned} \tag{31}$$

where the $\mathcal{L}V(x, t)$ is the generator which defines as

$$\mathcal{L}V(x, t) = \nabla V_t(x) \cdot u(X_t, t) + \frac{\varepsilon}{2} Trace\left[\nabla^2 V_t(x)\right] \tag{32}$$

Proof from SOC

Optimality Condition of SBP

Proof.

Recall the proof of HJB equation in the optimal control section. The key step is

$$
\begin{aligned}
V(s,\mathbf{z}) &= \inf_{\boldsymbol{\theta}} \left\{ \int_s^{s+\Delta s} L(t,\mathbf{x}(t),\boldsymbol{\theta}(t))\mathrm{d}t + V(s+\Delta s, \mathbf{x}(s+\Delta s)) \right\} \\
&\approx \inf_{\boldsymbol{\theta}} \left\{ L(s,\mathbf{z},\boldsymbol{\theta}(s))\Delta s + V(s+\Delta s, \mathbf{x}(s+\Delta s)) \right\} \\
&\approx \inf_{\boldsymbol{\theta}} \{ L(s,\mathbf{z},\boldsymbol{\theta}(s))\Delta s + V(s,\mathbf{x}(s)) \\
&\quad + \partial_s V(s,\mathbf{z})\Delta s + [\nabla_{\mathbf{z}} V(s,\mathbf{z})]^\top f(s,\mathbf{z},\boldsymbol{\theta}(s))\Delta s \} \\
\dot{\mathbf{x}}(t) &= f(t,\mathbf{x}(t),\boldsymbol{\theta}(t)), \quad t\in[s,\tau], \quad \mathbf{x}(s)=\mathbf{z}
\end{aligned}
\tag{31}
$$

Proof.

Similarly, we can derive the HJB equation for SOC as:

$$\begin{aligned} V(X_s, s) &= \inf_u \mathbb{E}\left\{\int_s^{s+\Delta s} w(X_t, u, t)\mathrm{d}t + V(X_{s+\Delta s}, s+\Delta s)\right\} \\ &\approx \inf_u \mathbb{E}\left\{w(X_s, u, s)\Delta s + V(X_{s+\Delta s}, s+\Delta s)\right\} \\ &\approx \inf_u \mathbb{E}\{w(X_t, u, t)\Delta s + V(X_s, s) \\ &\quad + \partial_s V(\mathbf{z}, s)\Delta s + \mathcal{L}V(\mathbf{z}, s)\Delta s + \nabla V_s(\mathbf{z}) \cdot \sqrt{\varepsilon}\Delta dW_s\} \\ \mathrm{d}X_t &= u(X_t, t)\,\mathrm{d}t + \sqrt{\varepsilon}\mathrm{d}W_t, \quad t \in [s, \tau], \quad X_s = \mathbf{z} \end{aligned} \tag{31}$$

Proof.

Then, the HJB equation for this problem is

$$\frac{\partial V(x,t)}{\partial t} + \min_{u\in\mathcal{U}} \left\{ \mathcal{L}V(x,t) + \frac{\|u(x,t)\|^2}{2\varepsilon} \right\} = 0, V(x,T) = 0. \tag{31}$$

Then, the optimal $u_t = -\varepsilon V_t$, substitute this optimal control, we can complete the proof. □

Theorem (Optimality Condition of SBP (Lagrange Function))

In the following, we give the optimality condition for SBP.

$$
\begin{aligned}
\frac{\partial \lambda}{\partial t} + \frac{1}{2}\|\nabla \lambda\|^2 + \frac{\varepsilon}{2}\Delta \lambda = 0 \\
\frac{\partial \mu_t}{\partial t} + \nabla \cdot (v_t \mu_t) - \frac{\varepsilon}{2}\Delta \mu_t = 0
\end{aligned}
\tag{32}
$$

where the optimal policy (control)

$$
v_t^* = \nabla \lambda(X_t) \tag{33}
$$

Proof.

We can also prove the same results from the Lagrange function. Consider the unconstrained minimization of the Lagrangian

$$\mathcal{L}(\mu, v) = \int_0^1 \int_{\mathbb{R}^n} \left[\frac{1}{2} \|v(t,x)\|^2 \mu_t(x) + \lambda(t,x) \left(\frac{\partial \mu_t}{\partial t} + \nabla \cdot (v \mu_t) - \frac{\varepsilon}{2} \Delta \mu_t \right) \right] \mathrm{d}x \, \mathrm{d}t \tag{34}$$

where μ_t satisfies the boundary condition. Then, integrating by parts, assuming that limits for $\|x\| \to \infty$ are zero, we get

Proof.

where μ_t satisfies the boundary condition. Then, integrating by parts, assuming that limits for $\|x\| \to \infty$ are zero, we get

$$
\begin{aligned}
\int_0^1 \int_{\mathbb{R}^n} & \left[\frac{1}{2}\|v(t,x)\|^2 - \left(\frac{\partial \lambda}{\partial t} + \nabla\lambda \cdot \nu + \frac{\varepsilon}{2}\Delta\lambda \right) \right] \mu_t(x) \mathrm{d}x \, \mathrm{d}t \\
& + \int_0^1 \int_{\mathbb{R}^n} \frac{\partial[\lambda(t,x)\mu_t(x)]}{\partial t} \mathrm{d}x \, \mathrm{d}t + \int_0^1 \int_{\mathbb{R}^n} \frac{\partial[\lambda(t,x)\nu(t,x)\mu_t(x)]}{\partial x} \mathrm{d}x \, \mathrm{d}t \qquad (34) \\
& - \frac{\varepsilon}{2} \int_0^1 \int_{\mathbb{R}^n} \frac{\partial[\lambda(t,x)\partial\mu_t(x)]}{\partial x} \mathrm{d}x \, \mathrm{d}t + \frac{\varepsilon}{2} \int_0^1 \int_{\mathbb{R}^n} \frac{\partial[\lambda(t,x)\mu_t(x)]}{\partial x} \mathrm{d}x \, \mathrm{d}t
\end{aligned}
$$

Proof.

The last four integrals are constant for a fixed λ and can therefore be discarded. Then, we consider doing this in two stages, starting from minimization with respect to ν for a fixed flow of probability densities μ_t. Pointwise minimization of the integral at each time gives that

$$\nu^*_{\mu_t}(t,x) = \nabla\lambda(t,x) \tag{34}$$

Then, substituting this expression for the optimal control, we obtain

$$J(\mu) = -\int_{\mathbb{R}^n}\int_0^1 \left[\frac{\partial\lambda}{\partial t} + \frac{1}{2}\|\nabla\lambda\|^2 + \frac{\varepsilon}{2}\Delta\lambda\right]\mu_t(x)\mathrm{d}t\,\mathrm{d}x \tag{35}$$

In view of this, if λ satisfies the Hamilton-Jacobi equation $\frac{\partial\lambda}{\partial t} + \frac{1}{2}\|\nabla\lambda\|^2 + \frac{\varepsilon}{2}\Delta\lambda = 0$, then $J(\mu)$ is identically zero. □

Theorem (Schrödinger System)

$$\begin{cases} \frac{\partial \Phi}{\partial t} = -\frac{\varepsilon}{2}\Delta \Phi \\ \frac{\partial \hat{\Phi}}{\partial t} = \frac{\varepsilon}{2}\Delta \hat{\Phi} \end{cases} \quad s.t. \quad \Phi(0,\cdot)\hat{\Phi}(0,\cdot) = \mu_0, \quad \Phi(1,\cdot)\hat{\Phi}(1,\cdot) = \mu_1. \tag{36}$$

Proof.

By applying the Hopf-Cole transform $(\lambda, \mu_t) \to (\Phi, \hat{\phi})$,

$$\Phi = \exp\left(\frac{\lambda}{\varepsilon}\right) \quad \text{and} \quad \hat{\Phi} = \mu_t \exp\left(\frac{-\lambda}{\varepsilon}\right), \tag{37}$$

1) For the first equation,

$$\begin{aligned} \frac{1}{\varepsilon} \exp\left(\frac{\lambda}{\varepsilon}\right) \frac{\partial \lambda}{\partial t} &= -\frac{1}{2\varepsilon} \exp\left(\frac{\lambda}{\varepsilon}\right) \|\nabla \lambda\|^2 - \frac{1}{2} \exp\left(\frac{\lambda}{\varepsilon}\right) \Delta\lambda \\ \frac{\partial \lambda}{\partial t} &= -\frac{1}{2}\|\nabla \lambda\|^2 - \frac{\varepsilon}{2}\Delta\lambda \end{aligned} \tag{38}$$

Proof.

2) For the second equation,

$$\begin{aligned}
\frac{\partial \mu_t}{\partial t} \exp\left(-\frac{\lambda}{\varepsilon}\right) + \mu_t \exp\left(-\frac{\lambda}{\varepsilon}\right)\left(-\frac{1}{\varepsilon}\right)\frac{\partial \lambda}{\partial t} \\
&= \frac{\varepsilon}{2}\frac{\partial}{\partial t}\left[\nabla \mu_t \exp\left(-\frac{\lambda}{\varepsilon}\right) + \mu_t \exp\left(-\frac{\lambda}{\varepsilon}\right)\left(-\frac{1}{\varepsilon}\right)\nabla\lambda\right] \\
&= \frac{\varepsilon}{2}\Delta\mu_t \exp\left(-\frac{\lambda}{\varepsilon}\right) + \frac{\varepsilon}{2}\nabla\mu_t \exp\left(-\frac{\lambda}{\varepsilon}\right)\left(-\frac{1}{\varepsilon}\right)\nabla\lambda \\
&+ \frac{\varepsilon}{2}\nabla\mu_t \exp\left(-\frac{\lambda}{\varepsilon}\right)\left(-\frac{1}{\varepsilon}\right)\nabla\lambda + \frac{\varepsilon}{2}\mu_t \exp\left(-\frac{\lambda}{\varepsilon}\right)\left(\frac{1}{\varepsilon^2}\right)\|\nabla\lambda\|^2 \\
&+ \frac{\varepsilon}{2}\mu_t \exp\left(-\frac{\lambda}{\varepsilon}\right)\left(-\frac{1}{\varepsilon}\right)\Delta\lambda
\end{aligned} \tag{37}$$

Proof.

$$\begin{aligned}\frac{\partial \mu_t}{\partial t} + \mu_t \left(-\frac{1}{\varepsilon}\right) \frac{\partial \lambda}{\partial t} = \\ \frac{\varepsilon}{2}\Delta \mu_t + \frac{\varepsilon}{2}\nabla \mu_t \left(-\frac{1}{\varepsilon}\right) \nabla \lambda + \frac{\varepsilon}{2}\nabla \mu_t \left(-\frac{1}{\varepsilon}\right) \nabla \lambda \\ + \frac{\varepsilon}{2}\mu_t \left(\frac{1}{\varepsilon^2}\right) \|\nabla \lambda\|^2 + \frac{\varepsilon}{2}\mu_t \left(-\frac{1}{\varepsilon}\right) \Delta \lambda\end{aligned} \tag{37}$$

substitute the equation $\frac{\partial \lambda}{\partial t} + \frac{1}{2}\|\nabla \lambda\|^2 + \frac{\varepsilon}{2}\Delta \lambda = 0$ into the above equation

Proof.

$$
\begin{aligned}
\frac{\partial \mu_t}{\partial t} + \mu_t \left(-\frac{1}{\varepsilon}\right) \frac{\partial \lambda}{\partial t} = \\
\frac{\partial \mu_t}{\partial t} + \frac{\mu_t}{\varepsilon}\left(\frac{1}{2}\|\nabla\lambda\|^2 + \frac{\varepsilon}{2}\Delta\lambda\right) &= \frac{\varepsilon}{2}\Delta\mu_t + \frac{\varepsilon}{2}\nabla\mu_t\left(-\frac{1}{\varepsilon}\right)\nabla\lambda + \frac{\varepsilon}{2}\nabla\mu_t\left(-\frac{1}{\varepsilon}\right)\nabla\lambda \\
&\quad + \frac{\varepsilon}{2}\mu_t\left(\frac{1}{\varepsilon^2}\right)\|\nabla\lambda\|^2 + \frac{\varepsilon}{2}\mu_t\left(-\frac{1}{\varepsilon}\right)\Delta\lambda \\
\frac{\partial \mu_t}{\partial t} &= \frac{\varepsilon}{2}\Delta\mu_t - \nabla\mu_t\nabla\lambda - \mu_t\Delta\lambda
\end{aligned}
\tag{37}
$$

which completes the proof.

Proof.

3) The Lagrangian function of static SBP has the form

$$\mathcal{L}(P_{01}, \lambda, \mu) = \int \int \log \left(\frac{P_{01}(x,y)}{W_{01}^{\varepsilon}(x,y)} \right) P_{01}(x,y) \, \mathrm{d}x \, \mathrm{d}y + \int \lambda(x) \left[\int P_{01}(x,y) \, \mathrm{d}y - \rho_0(x) \right] \mathrm{d}x + \int \mu(y) \left[\int P_{01}(x,y) \, \mathrm{d}x - \rho_1(y) \right] \mathrm{d}y \tag{37}$$

Setting the first variation equal to zero, we get the sufficient optimality condition

$$1 + \log \, P_{01}^{*}(x,y) - \log q_0(x) - \log p(0,x,1,y) + \lambda(x) + \mu(y) = 0 \tag{38}$$

where we have used the expression $W_{01}^{\varepsilon}(x,y) = q_0(x) \, p(0,x,1,y)$.

Proof.

Then, we get

$$
\begin{aligned}
\frac{P_{01}^{*}(x,y)}{p(0,x,1,y)} &= \exp\left[\log \rho_0^{W}(x) - 1 - \lambda(x) - \mu(y)\right] \\
&= \exp\left[\log \rho_0^{W}(x) - 1 - \lambda(x)\right] \exp\left[-\mu(y)\right] \\
&= \hat{\Phi}(x) + \Phi(y)
\end{aligned} \tag{37}
$$

Then, the optimal $P_{01}^{*}(x,y)$ has then the form

$$
P_{01}^{*}(x,y) = \hat{\Phi}(x)\, p(0,x,1,y)\, \Phi(y) \tag{38}
$$

Proof.

with Φ and $\hat{\Phi}$ satisfying

$$\hat{\Phi}(x)\int p(0,x,1,y)\,\Phi(y)\,\mathrm{d}y=\rho_0(x),\quad \Phi(y)\int p(0,x,1,y)\,\hat{\Phi}(x)\,\mathrm{d}x=\rho_1(y) \tag{37}$$

Let $\hat{\Phi}(0,x)=\hat{\Phi}(x),\Phi(1,y)=\Phi(y)$ and

$$\hat{\Phi}(1,y)=\int p(0,x,1,y)\,\hat{\Phi}(0,x)\,\mathrm{d}x,\quad \Phi(0,x)=\int p(0,x,1,y)\,\Phi(1,y)\,\mathrm{d}y \tag{38}$$

with the boundary conditions

$$\Phi(0,x)\cdot\hat{\Phi}(0,x)=\rho_0(x),\quad \Phi(1,y)\cdot\hat{\Phi}(1,y)=\rho_1(y). \tag{39}$$

□

Outline

Definition (Dynamic SBP)

$$P_{\text{SBP}} := arg \min_{P \in \mathscr{D}(\rho_0, \rho_1)} \mathbb{D}(P||\tilde{P}) \tag{40}$$

where $\hat{P}$ represents the prior path measure induced by the stochastic differential equation $\mathrm{d}X_t = f(t, X_t)\,\mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}W_t$ and

$$\mathbb{D}(P||\tilde{P}) = \mathbb{E}_P \left\{ \log \frac{\mathrm{d}P}{\mathrm{d}\tilde{P}} \right\}, \quad \text{if} P \ll \tilde{P} \tag{41}$$

denotes the relative entropy (KL divergence), and

$$\mathscr{D}(\rho_0, \rho_1) = \{P \in \mathcal{C}([0,1], \mathbb{R}^n) | P_{t=0} = \rho_0, P_{t=1} = \rho_1\} \tag{42}$$

denotes a path measure has marginal measure ρ_0 and ρ_1 at time $t = 0$ and $t = 1$, respectively.

Corollary (SOC-SBP)

$$
\begin{aligned}
&\inf_{v} \mathbb{E}\left\{\int_0^1 \frac{1}{2\varepsilon}\|v(t,X_t)\|^2 \,\mathrm{d}t\right\} \\
&\textit{s.t.} \quad \mathrm{d}X_t = [f(t,X_t) + v(t,X_t)]\,\mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}W_t, \quad X_0 \sim \mu_0, \quad X_1 \sim \mu_1,
\end{aligned}
\tag{43}
$$

Corollary (DynamicEOT-SBP)

Let

$$\tilde{v}(t,x) = f(t,x) - \frac{\varepsilon}{2}\nabla log \tilde{\mu}_t(t,x) \tag{44}$$

be the velocity of the prior process. We can also present a dynamic version for entropic optimal transport SBP as

$$\inf_{(\mu_t,v)} \int_0^1 \int_{\mathbb{R}^n} \left[\frac{1}{2\varepsilon}\|v(t,x) - \tilde{v}(t,x)\|^2 + \frac{\varepsilon}{8}\|\nabla log \frac{\mu(t,x)}{\tilde{\mu}(t,x)}\|^2 \right] \mathrm{d}\mu_t(x)\,\mathrm{d}t,$$
$$s.t.\ \frac{\partial \mu_t}{\partial t} + \nabla \cdot (v\mu_t) = 0, \quad \mu_{t=0} = \mu_0,\ \mu_{t=1} = \mu_1, \tag{45}$$

Theorem (Optimality Criteria)

let us consider a general Markovian prior measure $\tilde{P}$ which induced by a forward SDE $\mathrm{d}X_t = f(t, X_t)\,\mathrm{d}t + \sqrt{\varepsilon}\,\mathrm{d}W_t$. Then the corresponding optimality criteria defines as

$$\begin{cases} \frac{\partial \Phi}{\partial t} = -\frac{\varepsilon}{2}\Delta\Phi - f \cdot \nabla\Phi \\ \frac{\partial \hat{\Phi}}{\partial t} = \frac{\varepsilon}{2}\Delta\hat{\Phi} - \nabla \cdot (f\hat{\Phi}) \end{cases} \quad s.t. \quad \Phi(0,\cdot)\hat{\Phi}(0,\cdot) = \mu_0, \quad \Phi(1,\cdot)\hat{\Phi}(1,\cdot) = \mu_1. \tag{46}$$

Reference

- Machine-learning approaches for the empirical Schrodinger bridge problem
- Stochastic control liaisons: Richard Sinkhorn meets Gaspard Monge on a Schroedinger bridge
- Optimal Transport in Systems and Control
- On the relation between optimal transport and Schrödinger bridges: A stochastic control viewpoint